Linear Algebra II

07/04/2020, Tuesday, 15:00 – 18:30 (deadline for handing in: 18.30)

- This Take-Home Exam is 'open-book', which means that the book as well as lecture notes may be used as a reference.
- For handing in the exam, the use of electronic devices is of course allowed. The student is fully responsible for handing in his/her complete work before the deadline. You are asked to upload your answers as a **pdf-file**.
- Every student must upload the signed declaration before the start of the exam. An exam will not be graded in case the signed declaration has not been uploaded. After grading, short discussions with (a selection of) students will be held to check for possible fraud.
- Write your name and student number on each page!

1 $(4 + 7 + 7 = 18$ pts) **Least squares approximation**

Consider the real inner product space $\mathbb{R}^{2\times 2}$ of real 2×2 matrices with the inner product $\langle A, B\rangle = \mathrm{trace}(A^T B)$. Let $S \subset \mathbb{R}^{2\times 2}$ be the subset of all symmetric matrices.

(a) Show that S is a linear subspace.

(b) Determine an orthonormal basis of S.

(c) Compute the orthogonal projection of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ onto the subspace S.

2 $(4 + 4 + 6 + 4 = 18$ pts) **Eigenvalues**

Let A be a real $n \times n$ matrix and let $\mathcal{V}$ be an A-invariant subspace of $\mathbb{R}^n$. Suppose that $\dim(\mathcal{V}) = r < n$. Let V be an $n \times r$ matrix such that for the range of V we have $R(V) = \mathcal{V}$.

(a) Show that there exists a matrix $A_{11} \in \mathbb{R}^{r\times r}$ such that $AV = VA_{11}$.

(b) Prove that every eigenvalue of A_{11} is an eigenvalue of A.

(c) Prove that there exists a basis β of $\mathbb{R}^n$ such that the matrix of A with respect to β has the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11} is the matrix obtained in part (a) of this problem.

(d) Prove that the characteristic polynomial of A_{11} divides the characteristic polynomial of A.

3 $(3+5+5+5=18 \text{ pts})$ **Diagonalization**

We say that two $n \times n$ matrices are *simultaneously diagonalizable* if there exists a non-singular $n \times n$ matrix S such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal (not necessarily identical).

(a) Let I denote the $n \times n$ identity matrix and let A be any diagonalizable $n \times n$ matrix. Show that I and A are simultaneously diagonalizable.

(b) Show that if A and B are simultaneously diagonalizable then $AB = BA$.

(c) Let D be a diagonal $n \times n$ matrix with n distinct entries on the diagonal. Find all $n \times n$ matrices that commute with D.

(d) Show that if $AB = BA$ and A has n distinct eigenvalues, then A and B are simultaneously diagonalizable.

4 $(4+4+5+5=18 \text{ pts})$ **Positive definite matrices**

Let A be a real symmetric $n \times n$ matrix

(a) Suppose that B is a nonsingular $n \times n$ matrix. Prove that A is positive definite if and only if B^TAB is positive definite.

Now suppose that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$$

with A_{11} and A_{22} symmetric matrices.

(b) Prove that if A is positive definite then A_{11} and A_{22} are positive definite.

(c) Assume that A_{11} is nonsingular. Determine a matrix X such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{12}^T A_{11}^{-1} A_{12} \end{pmatrix} \begin{pmatrix} I & X^T \\ 0 & I \end{pmatrix}$$

(d) Prove that $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$ is positive definite if and only if A_{11} and $A_{22} - A_{12}^T A_{11}^{-1} A_{12}$ are positive definite.

5 (14 + 4 = 18 pts) **Singular value decomposition**

Let

$$M = \begin{pmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{pmatrix}$$

where a, b, and c real numbers with $a > b > c$.

(a) Find a singular value decomposition of M.

(b) Find the best rank 2 approximation of M.

6 (5 + 5 + 8 = 18 pts) **Jordan Form**

Let $A \in \mathbb{C}^{5\times 5}$.

(a) Assume A has three distinct eigenvalues, λ_1, λ_2 and λ_3, with geometric multiplicities $g_1 = 2$, $g_2 = 1$ and $g_3 = 1$, respectively. Assume the characteristic polynomial is $p_A(z) = (z - \lambda_1)^2(z - \lambda_2)^2(z - \lambda_3)$. Determine the Jordan Form. Motivate your answer

(b) Determine the minimal polynomial of the matrix A specified in part (a). Motivate your answer.

(c) Assume now A has two distinct eigenvalues, λ_1 and λ_2. Assume its minimal polynomial is $(z - \lambda_1)(z - \lambda_2)^2$. Assume that the algebraic multiplicity of λ_1 is $a_1 = 1$. Determine all possible Jordan Forms of A. Motivate your answer.

10 pts free